

A REMARK ON THE CONSTRUCTIBILITY OF REAL ROOT REPRESENTATIONS OF QUIVERS USING UNIVERSAL EXTENSION FUNCTORS

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ABSTRACT. In this paper we consider the following question: Is it possible to construct all real root representations of a given quiver Q by using universal extension functors, starting with a real Schur representation? We give a concrete example answering this question negatively.

0. INTRODUCTION

Let k be a field and let Q be a (finite) quiver. We fix a representation S with $\text{End}_{kQ} S = k$ and $\text{Ext}_{kQ}^1(S, S) = 0$. In analogy to [3, Section 1] we consider the following subcategories of $\text{rep}_k Q$. Let \mathfrak{M}^S be the full subcategory of all modules X with $\text{Ext}_{kQ}^1(S, X) = 0$ such that, in addition, X has no direct summand which can be embedded into some direct sum of copies of S . Similarly, let \mathfrak{M}_S be the full subcategory of all modules X with $\text{Ext}_{kQ}^1(X, S) = 0$ such that, in addition, no direct summand of X is a quotient of a direct sum of copies of S . Finally, let \mathfrak{M}^{-S} be the full subcategory of all modules X with $\text{Hom}_{kQ}(X, S) = 0$, and let \mathfrak{M}_{-S} be the full subcategory of all modules X with $\text{Hom}_{kQ}(S, X) = 0$. Moreover, we consider

$$\mathfrak{M}_S^S = \mathfrak{M}^S \cap \mathfrak{M}_S, \quad \mathfrak{M}_{-S}^{-S} = \mathfrak{M}^{-S} \cap \mathfrak{M}_{-S}.$$

According to [3, Proposition 1 & 1* and Proposition 2], we have the following equivalences of categories

$$\begin{aligned} \overline{\sigma}_S &: \mathfrak{M}^{-S} \rightarrow \mathfrak{M}^S/S, \\ \underline{\sigma}_S &: \mathfrak{M}_{-S} \rightarrow \mathfrak{M}_S/S, \\ \sigma_S &: \mathfrak{M}_{-S}^{-S} \rightarrow \mathfrak{M}_S^S/S, \end{aligned}$$

where \mathfrak{M}^S/S denotes the quotient category of \mathfrak{M}^S modulo the maps which factor through direct sums of copies of S , similarly for \mathfrak{M}_S/S and \mathfrak{M}_S^S/S . We call the functor σ_S *universal extension functor*. A brief description of these functors is given in Section 1. This paper is dedicated to the following question.

Question (\star). *Let α be a positive non-Schur real root for Q and let X_α be the unique indecomposable representation of dimension vector α .*

Does there exist a sequence of real Schur roots β_1, \dots, β_n ($n \geq 2$) such that

$$X_\alpha = \sigma_{X_{\beta_n}} \cdot \dots \cdot \sigma_{X_{\beta_2}}(X_{\beta_1}) \quad ?$$

Here, X_{β_i} denotes the unique indecomposable representation of dimension vector β_i .

One might reformulate the above question as follows. Is it possible to construct all real root representations of Q using universal extension functors, starting with a real Schur representation?

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One of the nice facts about the universal extension functor σ_S is that it allows one to keep track of certain properties of representations. For instance, the functor σ_S preserves indecomposable tree representations [7, Lemma 3.16] (for a definition of “tree representation” and background results we refer the reader to [4, Introduction]) and, moreover, if we apply the functor σ_S to a representation of known endomorphism ring dimension, we can easily compute the dimension of the endomorphism ring of the resulting representation [3, Proposition 3 & 3*]. Hence, if $X_\alpha = \sigma_{X_{\beta_n}} \cdot \dots \cdot \sigma_{X_{\beta_2}}(X_{\beta_1})$ with β_i ($i = 1, \dots, n$) real Schur roots, then X_α is a tree representation and one can easily compute $\dim \text{End}_{kQ} X_\alpha$.

Question (\star) was first answered affirmatively by Ringel [3, Section 2] for the quiver

$$Q(g, h) : 1 \begin{array}{c} \xrightarrow{\mu_1} \\ \vdots \\ \xrightarrow{\mu_g} \\ \vdots \\ \xrightarrow{\nu_1} \\ \vdots \\ \xrightarrow{\nu_h} \end{array} 2,$$

with $g, h \geq 1$. In [7, Theorem B] Question (\star) was answered affirmatively for the quiver

$$Q(f, g, h) : 1 \begin{array}{c} \xrightarrow{l_1} \\ \vdots \\ \xrightarrow{l_f} \end{array} 2 \begin{array}{c} \xrightarrow{\mu_1} \\ \vdots \\ \xrightarrow{\mu_g} \\ \vdots \\ \xrightarrow{\nu_1} \\ \vdots \\ \xrightarrow{\nu_h} \end{array} 3,$$

with $f, g, h \geq 1$. More examples of real root representations which can be constructed using universal extension functors can be found in [8, Appendix].

Hence, there are quivers for which Question (\star) can be answered affirmatively. The question is, can it be answered affirmatively in general? Unfortunately the answer is negative in general.

Answer (to Question (\star)). *In Section 2 we give a concrete example answering Question (\star) negatively.*

This paper is organized as follows. In Section 1 we discuss further notation and background results and in Section 2 we describe an example answering Question (\star) negatively.

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1. FURTHER NOTATION AND BACKGROUND RESULTS

Let k be a field. Let Q be a finite quiver, i.e. an oriented graph with finite vertex set Q_0 and finite arrow set Q_1 together with two functions $h, t : Q_1 \rightarrow Q_0$ assigning head and tail to each arrow $a \in Q_1$. A representation X of Q is given by a vector space X_i (over k) for each vertex $i \in Q_0$ together with a linear map $X_a : X_{t(a)} \rightarrow X_{h(a)}$ for each arrow $a \in Q_1$. Let X and Y be two representations of Q . A homomorphism $\phi : X \rightarrow Y$ is given by linear maps $\phi_i : X_i \rightarrow Y_i$ such that for each arrow $a \in Q_1$, $a : i \rightarrow j$ say, the square

$$\begin{array}{ccc} X_i & \xrightarrow{X_a} & X_j \\ \phi_i \downarrow & & \downarrow \phi_j \\ Y_i & \xrightarrow{Y_a} & Y_j \end{array}$$

commutes.

A dimension vector for Q is given by an element of \mathbb{N}^{Q_0} . We will write e_i for the coordinate vector at vertex i and by $\alpha[i]$, $i \in Q_0$, we denote the i -th coordinate of $\alpha \in \mathbb{N}^{Q_0}$. We can partially order \mathbb{N}^{Q_0} via $\alpha \geq \beta$ if $\alpha[i] \geq \beta[i]$ for all $i \in Q_0$. We define $\alpha > \beta$ to mean $\alpha \geq \beta$ and $\alpha \neq \beta$. If X is a finite dimensional representation, meaning that all vector spaces X_i ($i \in Q_0$) are finite dimensional, then $\underline{\dim} X = (\dim X_i)_{i \in Q_0}$ is the dimension vector of X . Throughout this paper we only consider finite dimensional representations. We denote by $\mathbf{rep}_k Q$ the full subcategory with objects the finite dimensional representations of Q . The Ringel form on \mathbb{Z}^{Q_0} is defined by

$$\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha[i] \beta[i] - \sum_{a \in Q_1} \alpha[t(a)] \beta[h(a)]$$

Moreover, let $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$ be its symmetrization.

We say that a vertex $i \in Q_0$ is loop-free if there are no arrows $a : i \rightarrow i$. By a quiver without loops we mean a quiver with only loop-free vertices. For a loop-free vertex $i \in Q_0$ the simple reflection $s_i : \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}^{Q_0}$ is defined by

$$s_i(\alpha) := \alpha - (\alpha, e_i) e_i.$$

A simple root is a vector e_i for $i \in Q_0$. The set of simple roots is denoted by Π . The Weyl group, denoted by W , is the subgroup of $\mathrm{GL}(\mathbb{Z}^n)$, where $n = |Q_0|$, generated by the s_i . By $\Delta_{\mathrm{re}}^+(Q) := \{\alpha \in W(\Pi) : \alpha > 0\}$ we denote the set of (positive) real roots for Q .

We have the following remarkable theorem.

Theorem 1.1 (Kac [2, Theorem 1 and 2], Schofield [6, Theorem 9]). *Let k be a field, Q be a quiver and let $\alpha \in \Delta_{\mathrm{re}}^+(Q)$. There exists a unique indecomposable representation (up to isomorphism) of dimension vector α .*

For finite fields and algebraically closed fields the theorem is due to Kac [2, Theorem 1 and 2]. As pointed out in the introduction of [6], Kac's method of proof showed that the above theorem holds for fields of characteristic p . The proof for fields of characteristic zero is due to Schofield [6, Theorem 9].

For a given positive real root α for Q the unique indecomposable representation (up to isomorphism) of dimension vector α is denoted by X_α . By a real root representation we mean an X_α for α a positive real root. A Schur representation is a representation with $\mathrm{End}_{kQ}(X) = k$. By a real Schur representation we mean a real representation which is also a Schur representation. A positive real root is called a real Schur root if X_α is a real Schur representation.

We have the following useful formula: if X, Y are representations of Q then we have

$$\dim \mathrm{Hom}_{kQ}(X, Y) - \dim \mathrm{Ext}_{kQ}^1(X, Y) = \langle \underline{\dim} X, \underline{\dim} Y \rangle.$$

It follows that $\mathrm{Ext}_{kQ}^1(X_\alpha, X_\alpha) = 0$ for α a real Schur root.

1.1. Universal Extension Functors. We use this section to describe briefly how the functors

$$\begin{aligned} \overline{\sigma}_S & : \mathfrak{M}^{-S} \rightarrow \mathfrak{M}^S/S, \\ \underline{\sigma}_S & : \mathfrak{M}_{-S} \rightarrow \mathfrak{M}_S/S, \\ \sigma_S & : \mathfrak{M}_{-S}^{-S} \rightarrow \mathfrak{M}_S^S/S, \end{aligned}$$

operate on objects.

The functor $\overline{\sigma}_S$ is given by the following construction: Let $X \in \mathfrak{M}^{-S}$ and let E_1, \dots, E_r be a basis of the k -vector space $\text{Ext}_{kQ}^1(S, X)$. Consider the exact sequence E given by the elements E_1, \dots, E_r

$$E : 0 \rightarrow X \rightarrow Z \rightarrow \bigoplus_r S \rightarrow 0.$$

According to [3, Lemma 3] we have $Z \in \mathfrak{M}^S$ and we define $\overline{\sigma}_S(X) := Z$. Now, let $Y \in \mathfrak{M}_{-S}$ and let E'_1, \dots, E'_s be a basis of the k -vector space $\text{Ext}_{kQ}^1(Y, S)$. Consider the exact sequence E' given by E'_1, \dots, E'_s

$$E' : 0 \rightarrow \bigoplus_s S \rightarrow U \rightarrow Y \rightarrow 0.$$

Then we have $U \in \mathfrak{M}_S$ and we set $\underline{\sigma}_S(Y) := U$. The functor σ_S is given by applying both constructions successively.

The inverse $\overline{\sigma}_S^{-1}$ is constructed as follows: Let $X \in \mathfrak{M}^S$ and let ϕ_1, \dots, ϕ_r be a basis of the k -vector space $\text{Hom}_{kQ}(X, S)$. Then by [3, Lemma 2] the sequence

$$0 \rightarrow X^{-S} \rightarrow X \xrightarrow{(\phi_i)_i} \bigoplus_r S \rightarrow 0$$

is exact, where X^{-S} denotes the intersection of the kernels of all maps $X \rightarrow S$. We set $\overline{\sigma}_S^{-1}(X) := X^{-S}$. Now, let $Y \in \mathfrak{M}_S$. The inverse $\underline{\sigma}_S^{-1}$ is given by $\underline{\sigma}_S^{-1}(Y) := Y/Y'$, where Y' is the sum of the images of all maps $S \rightarrow Y$. The inverse σ_S^{-1} is given by applying both constructions successively.

Both constructions show that

$$(\dagger) \quad \dim \sigma_S^{\pm 1}(X) = \dim X - (\dim X, \dim S) \cdot \dim S.$$

Moreover, we have the following proposition.

Proposition 1.2 ([3, Proposition 3 & 3*]). *Let $X \in \mathfrak{M}_{-S}^{-S}$. Then*

$$\dim \text{End}_{kQ} \sigma_S(X) = \dim \text{End}_{kQ}(X) + \langle \dim X, \dim S \rangle \cdot \langle \dim S, \dim X \rangle.$$

Let $Y \in \mathfrak{M}_S^S$. Then

$$\dim \text{End}_{kQ} \sigma_S^{-1}(Y) = \dim \text{End}_{kQ}(Y) - \langle \dim Y, \dim S \rangle \cdot \langle \dim S, \dim Y \rangle.$$

2. A NEGATIVE AND UNPLEASANT EXAMPLE

Let k be a field and let Q be a quiver. We recall Question (\star) stated in the introduction.

Question (\star) . *Let α be a positive non-Schur real root for Q and let X_α be the unique indecomposable representation of dimension vector α .*

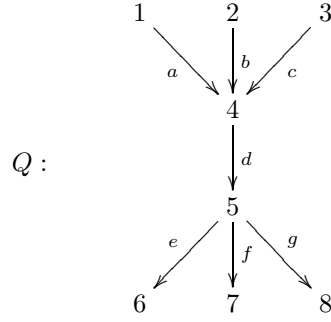
Does there exist a sequence of real Schur roots β_1, \dots, β_n ($n \geq 2$) such that

$$X_\alpha = \sigma_{X_{\beta_n}} \cdot \dots \cdot \sigma_{X_{\beta_2}}(X_{\beta_1}) \quad ?$$

We remark that in the case that X_α can be constructed in the above way we have $\beta_i < \alpha$ for $i = 1, \dots, n$.

In the following we give an explicit example of a non-Schur real root representations which cannot be constructed using universal extension functors.

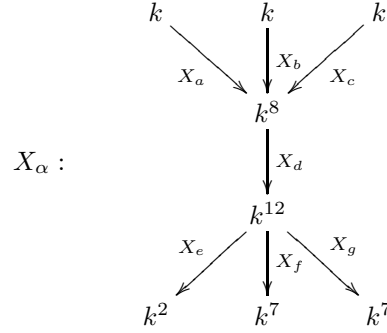
We consider the quiver Q



and the real root $\alpha = (1, 1, 1, 8, 12, 2, 7, 7) = s_8 s_7 s_5 s_4 s_8 s_7 s_5 s_8 s_7 s_5 s_6 s_4 s_5 s_4 s_1 s_2 s_3(e_4)$.

For the convenience of the reader we give an explicit description of the representation X_α .

We start by considering the representation X_α over the field $k = \mathbb{Q}$. In this case, one can use the result [1, Proposition A.4] to construct the representation X_α ; we get



with

$$X_a = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^t,$$

$$X_b = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^t,$$

$$X_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}^t,$$

$$X_d = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{aligned}
X_e &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}, \\
X_f &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \\
X_g &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

In particular, we see that X_α is a tree representation.

The representation X_α , as given above, is defined over every field k . Moreover, it is not difficult to see that $\text{End}_{kQ}(X_\alpha)$ is local. Hence, the representation X_α is the unique indecomposable representation of dimension vector α over every field k .

Moreover, $\dim \text{End}_{kQ}(X_\alpha) = 9$ so that X_α is not a real Schur representation.

Theorem 2.1. *There exists no real Schur root β with the following properties:*

- (i) $X_\alpha \in \mathfrak{M}_{X_\beta}^{X_\beta}$, and
- (ii) $\text{Hom}_{kQ}(X_\alpha, X_\beta) \neq 0$ or $\text{Hom}_{kQ}(X_\beta, X_\alpha) \neq 0$.

If we had a sequence of real Schur roots β_1, \dots, β_n ($n \geq 2$) such that $X_\alpha = \sigma_{X_{\beta_n}} \cdot \dots \cdot \sigma_{X_{\beta_2}}(X_{\beta_1})$ then β_n would have to satisfy conditions (i) and (ii). Note that condition (ii) merely states that $\sigma_{X_{\beta_n}}^{-1}(X_\alpha) \neq X_\alpha$. Thus, once we have established the claim it is clear that X_α provides an example which answers Question (\star) negatively.

We use the rest of this section to prove the above theorem. We show that there are no real Schur roots satisfying (i).

Proof of Theorem 2.1. Condition (i) requires $\beta < \alpha$ by [3, Lemma 2] and

$$\text{Ext}_{kQ}^1(X_\alpha, X_\beta) = 0 = \text{Ext}_{kQ}^1(X_\beta, X_\alpha),$$

which implies that $\langle \alpha, \beta \rangle \geq 0$ and $\langle \beta, \alpha \rangle \geq 0$. Hence, we start by determining the set of real roots β with the following properties:

- (i') $\beta < \alpha$,
- (ii') $\langle \alpha, \beta \rangle \geq 0$ and $\langle \beta, \alpha \rangle \geq 0$.

These roots are potential candidates for a reflection. Using the arguments given in [5, Section 6], it is easy to determine the real roots β which satisfy (i') and (ii'): both conditions imply that $s_\alpha(\beta) < 0$ and, hence, if $s_\alpha = s_{i_1} \dots s_{i_n}$ we get $s_\alpha(\beta) = s_{i_1} \dots s_{i_n}(\beta) < 0$ if and only if $\beta = s_{i_n} \dots s_{i_{m+1}}(e_{i_m})$ for some m . Thus, once we have written s_α as a product of the generators s_i it is straightforward to find the real roots β satisfying (i') and (ii'). A decomposition of s_α into a product of the generators s_i can be achieved as follows: if $s_i(\alpha) = \alpha' < \alpha$ then $s_\alpha = s_i s_{\alpha'} s_i$; this gives an algorithm to find a shortest expression of s_α in terms of the s_i .

Applying the above algorithm to the real root α , we get the following potential candidates for a reflection

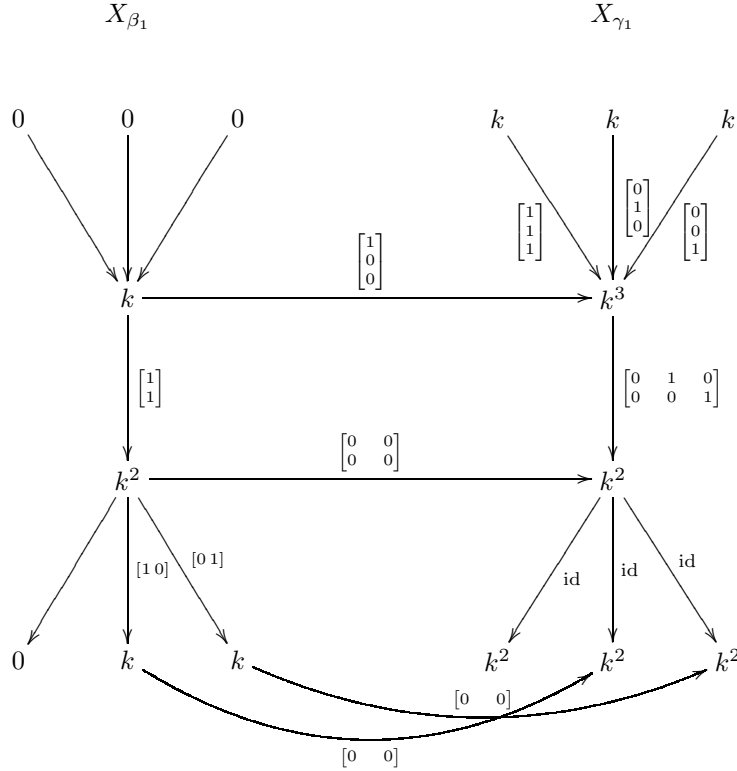
$$\begin{aligned}\beta_1 &= (0, 0, 0, 1, 2, 0, 1, 1), \\ \beta_2 &= (0, 1, 1, 4, 7, 1, 4, 4), \\ \beta_3 &= (1, 0, 1, 4, 7, 1, 4, 4), \quad \text{and} \\ \beta_4 &= (1, 1, 0, 4, 7, 1, 4, 4).\end{aligned}$$

We see that $\langle \beta_i, \alpha \rangle = 0 = \langle \alpha, \beta_i \rangle$ for $i = 2, 3, 4$, and hence the only reflection candidate is β_1 . Note that β_1 is a real Schur root, and hence indeed a candidate for a reflection. However, β_1 does not satisfy condition (i), that is $X_\alpha \notin \mathfrak{M}_{X_{\beta_1}}^{X_{\beta_1}}$.

Assume to the contrary that $X_\alpha \in \mathfrak{M}_{X_{\beta_1}}^{X_{\beta_1}}$. Then $\sigma_{X_{\beta_1}}^{-1}(X_\alpha) \in \mathfrak{M}_{-X_{\beta_1}}^{-X_{\beta_1}}$, that is

$$\text{Hom}_{kQ}(\sigma_{X_{\beta_1}}^{-1}(X_\alpha), X_{\beta_1}) = 0 = \text{Hom}_{kQ}(X_{\beta_1}, \sigma_{X_{\beta_1}}^{-1}(X_\alpha)).$$

Using formula (\dagger) from Section 1.1, we get $\gamma_1 := \underline{\dim} \sigma_{X_{\beta_1}}^{-1}(X_\alpha) = (1, 1, 1, 3, 2, 2, 2, 2)$. The following diagram, however, shows that $\text{Hom}_{kQ}(X_{\beta_1}, X_{\gamma_1}) \neq 0$. The representation X_{γ_1} can be constructed using the result [1, Proposition A.4] together with the same reasoning as for X_α to pass to any field k .



This is a contradiction, and hence $X_\alpha \notin \mathfrak{M}_{X_{\beta_1}}^{X_{\beta_1}}$ which completes the proof of the theorem and we see that, indeed, the representation X_α answers Question (\star) negatively. \square

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